A RELATIONS AND FUNCTIONS

The charges for parking a car in a short-term car park at an Airport are given in the table shown alongside.

Car park charges	
Period (h)	*Charge*
0 - 1 hours	\$5.00
1 - 2 hours	\$9.00
2 - 3 hours	\$11.00
3 - 6 hours	\$13.00
6 - 9 hours	\$18.00
9 - 12 hours	\$22.00
12 - 24 hours	\$28.00

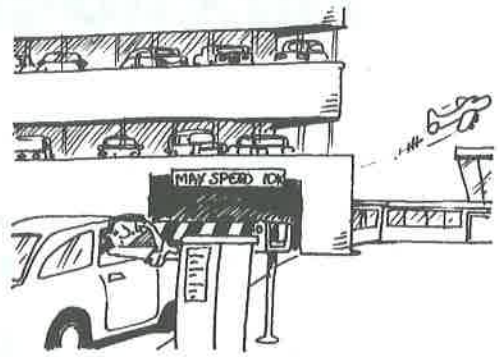
There is an obvious relationship between time spent and the cost. The cost is dependent on the length of time the car is parked.

Looking at this table we might ask: How much would be charged for exactly one hour? Would it be \$5 or \$9?

To make the situation clear, and to avoid confusion, we could adjust the table and draw a graph. We need to indicate that 2-3 hours really means for time over 2 hours up to and including 3 hours i.e., $2 < t \leqslant 3$.

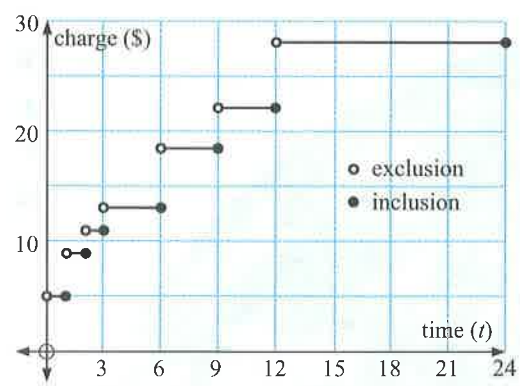
So, we now have

Car park charges	
Period	*Charge*
$0 < t \leqslant 1$ hours	\$5.00
$1 < t \leqslant 2$ hours	\$9.00
$2 < t \leqslant 3$ hours	\$11.00
$3 < t \leqslant 6$ hours	\$13.00
$6 < t \leqslant 9$ hours	\$18.00
$9 < t \leqslant 12$ hours	\$22.00
$12 < t \leqslant 24$ hours	\$28.00



In mathematical terms, because we have a relationship between two variables, time and cost, the schedule of charges is an example of a **relation**.

A relation may consist of a finite number of ordered pairs, such as $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ or an infinite number of ordered pairs.



The parking charges example is clearly the latter as any real value of time (t hours) in the interval $0 < t \leqslant 24$ is represented.

The set of possible values of the variable on the horizontal axis is called the **domain** of the relation.

For example:
- $\{t: \ 0 < t \leqslant 24\}$ is the domain for the car park relation
- $\{-2, 1, 4\}$ is the domain of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$.

The set which describes the possible y-values is called the **range** of the relation.

For example:
- the range of the car park relation is $\{5, 9, 11, 13, 18, 22, 28\}$
- the range of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ is $\{3, 5, 6\}$.

We will now look at relations and functions more formally.

RELATIONS

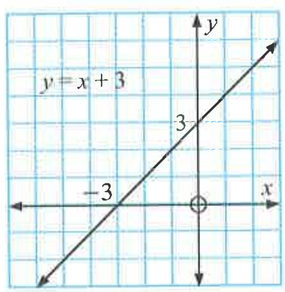
A **relation** is any set of points on the Cartesian plane.

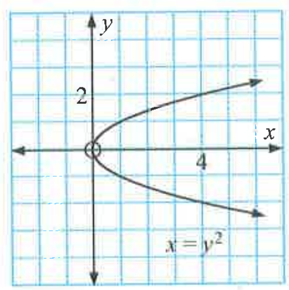
A relation is often expressed in the form of an **equation** connecting the **variables** x and y.

For example $y = x + 3$ and $x = y^2$ are the equations of two relations.

These equations generate sets of ordered pairs.

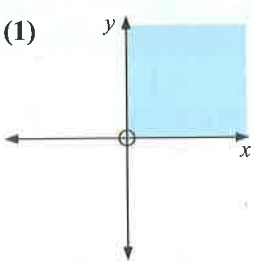
Their graphs are:





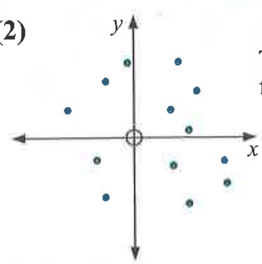
However, a relation may not be able to be defined by an equation. Below are two examples which show this:

(1)



All points in the first quadrant are a relation. $x > 0, y > 0$

(2)



These 13 points form a relation.

FUNCTIONS

A **function** is a relation in which no two different ordered pairs have the same x-coordinate (first member).

We can see from the above definition that a function is a special type of relation.

TESTING FOR FUNCTIONS

Algebraic Test:

If a relation is given as an equation, and the substitution of any value for x results in one and only one value of y, we have a function.

For example:
- $y = 3x - 1$ is a function, as for any value of x there is only one value of y
- $x = y^2$ is not a function since if $x = 4$, say, then $y = \pm 2$.

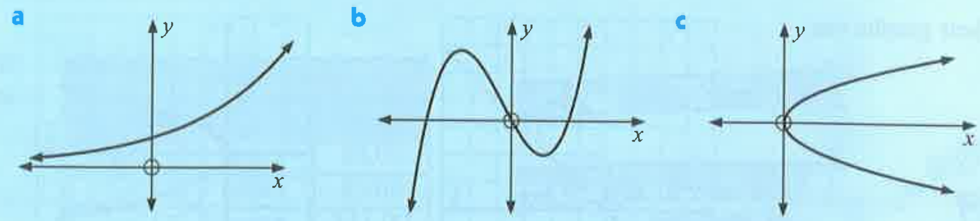
Geometric Test ("Vertical Line Test"):

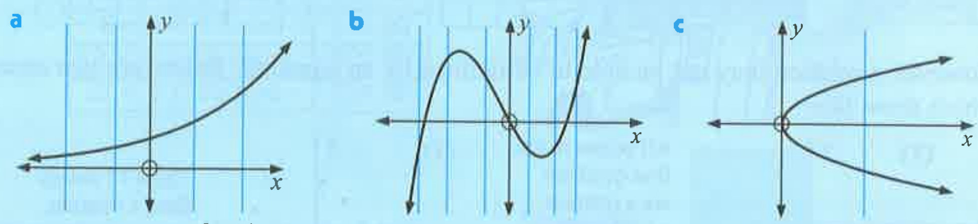
> If we draw all possible vertical lines on the graph of a relation, the relation:
> - is a function if each line cuts the graph no more than once
> - is not a function if one line cuts the graph more than once.

DEMO

Example 1

Which of the following relations are functions?





a: a function   b: a function   c: not a function

GRAPHICAL NOTE

- If a graph contains a small **open circle** end point such as ——o , the end point is **not included**.
- If a graph contains a small **filled-in circle** end point such as ——• , the end point **is included**.
- If a graph contains an **arrow head** at an end such as ——▸ then the graph continues indefinitely in that general direction, or the shape may repeat as it has done previously.

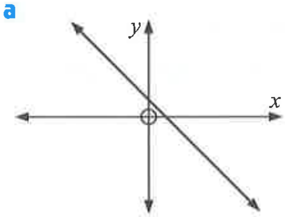
EXERCISE 1A

1 Which of the following sets of ordered pairs are functions? Give reasons.

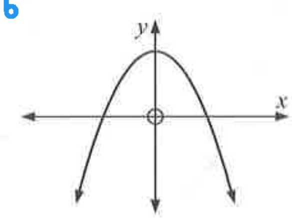
- **a** (1, 3), (2, 4), (3, 5), (4, 6)
- **b** (1, 3), (3, 2), (1, 7), (−1, 4)
- **c** (2, −1), (2, 0), (2, 3), (2, 11)
- **d** (7, 6), (5, 6), (3, 6), (−4, 6)
- **e** (0, 0), (1, 0), (3, 0), (5, 0)
- **f** (0, 0), (0, −2), (0, 2), (0, 4)

2 Use the vertical line test to determine which of the following relations are functions:

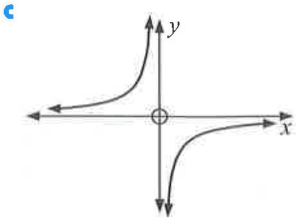
a



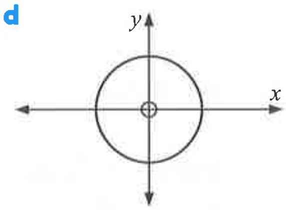
b



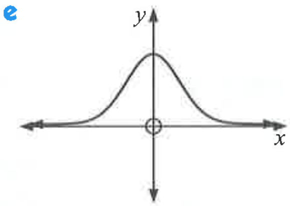
c



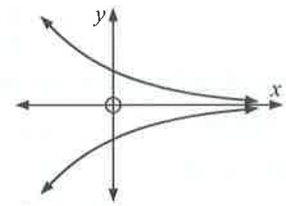
d



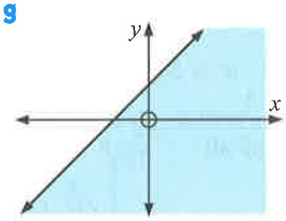
e



f



g



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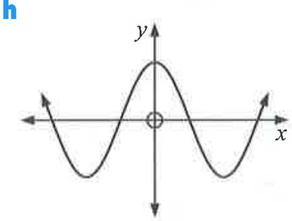
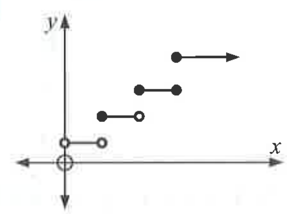


i



3 Will the graph of a straight line always be a function? Give evidence.

4 Give algebraic evidence to show that the relation $x^2 + y^2 = 9$ is not a function.

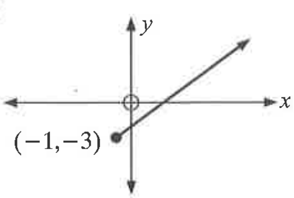
B INTERVAL NOTATION, DOMAIN AND RANGE

DOMAIN AND RANGE

The **domain** of a relation is the set of permissible values that x may have.
The **range** of a relation is the set of permissible values that y may have.

For example:

(1)



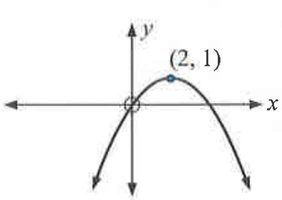
All values of $x \geqslant -1$ are permissible.

So, the domain is $\{x\colon\ x \geqslant -1\}$.

All values of $y \geqslant -3$ are permissible.

So, the range is $\{y\colon\ y \geqslant -3\}$.

(2)



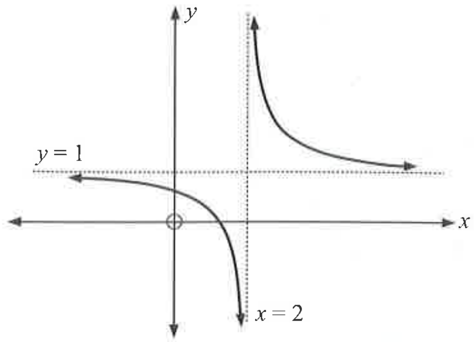
x can take any value.

So, the domain is $\{x\colon\ x \text{ is in } \mathcal{R}\}$.

y cannot be > 1

$\therefore$ range is $\{y\colon\ y \leqslant 1\}$.

(3)



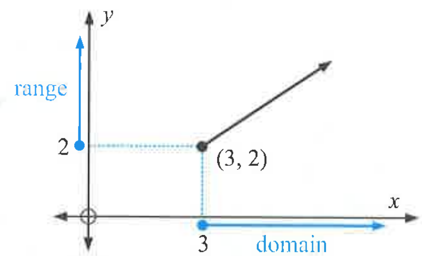
x can take all values except $x = 2$.

So, the domain is $\{x\colon\ x \neq 2\}$.

Likewise, the range is $\{y\colon\ y \neq 1\}$.

The domain and range of a relation are best described where appropriate using **interval notation**.

For example:

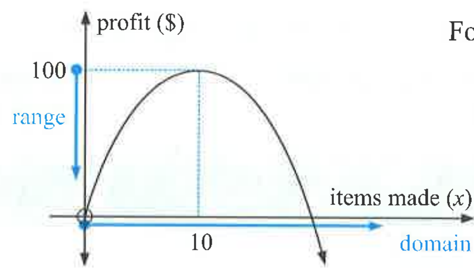


The domain consists of all real x such that $x \geqslant 3$ and we write this as

$\{x : x \geqslant 3\}$

the set of all   such that

Likewise the range would be $\{y\colon\ y \geqslant 2\}$.



For this profit function:

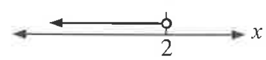
- the domain is $\{x\colon\ x \geqslant 0\}$
- the range is $\{y\colon\ y \leqslant 100\}$.

Intervals have corresponding graphs.

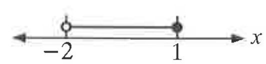
For example:

$\{x\colon\ x \geqslant 3\}$ or $[3, \infty[$ is read "the set of all x such that x is greater than or equal to 3" and has number line graph

$\{x\colon\ x < 2\}$ or $]-\infty, 2[$ has number line graph



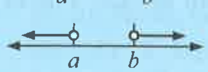
$\{x\colon\ -2 < x \leqslant 1\}$ or $]-2, 1]$ has number line graph



$\{x\colon\ x \leqslant 0$ or $x > 4\}$
i.e., $]-\infty, 0]$ or $]4, \infty[$ has number line graph

Note:

for numbers *between* a and b we write $a < x < b$ or $]a, b[$.



for numbers '*outside*' a and b we write $x < a$ or $x > b$
i.e., $]-\infty, a[$ or $]b, \infty[$.

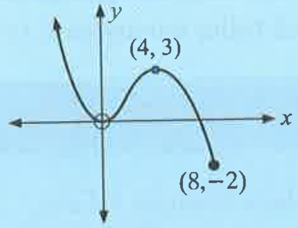
of x.

g

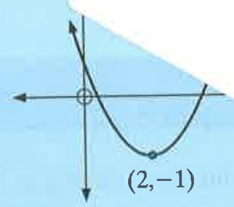
Example 2

For each of the following graphs state the domain and

a



b



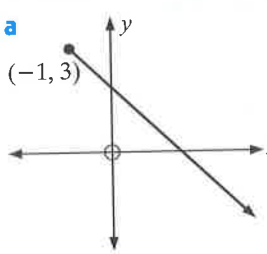
a Domain is $\{x\colon x \leqslant 8\}$.
Range is $\{y\colon y \geqslant -2\}$.

b Domain is $\{x\colon x \text{ is in } \mathcal{R}\}$.
Range is $\{y\colon y \geqslant -1\}$.

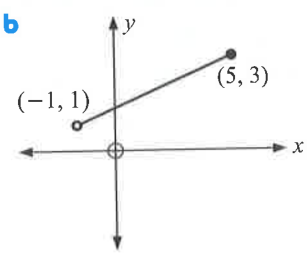
EXERCISE 1B

1 For each of the following graphs find the domain and range:

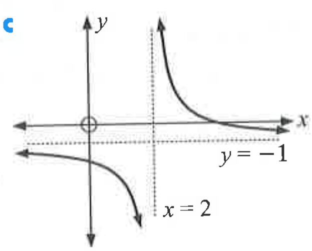
a



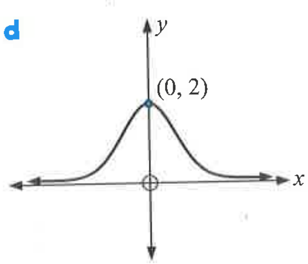
b



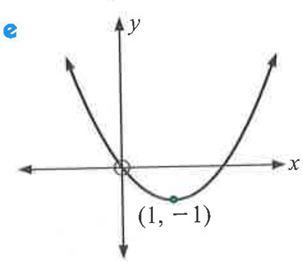
c



d



e



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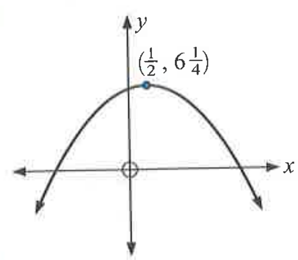

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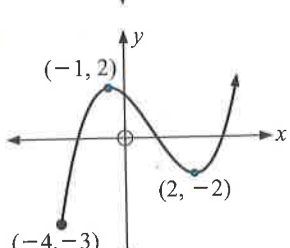

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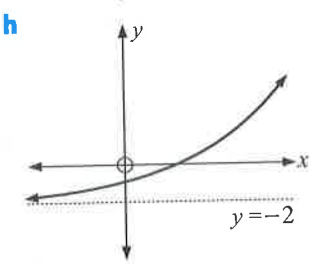

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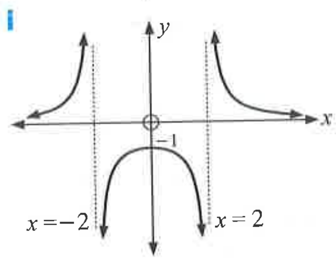


2 Use a graphics calculator to help sketch carefully the graphs of the following functions and find the domain and range of each:

a $f(x) = \sqrt{x}$ **b** $f(x) = \dfrac{1}{x^2}$ **c** $f(x) = \sqrt{4-x}$

d $y = x^2 - 7x + 10$ **e** $y = 5x - 3x^2$ **f** $y = x + \dfrac{1}{x}$

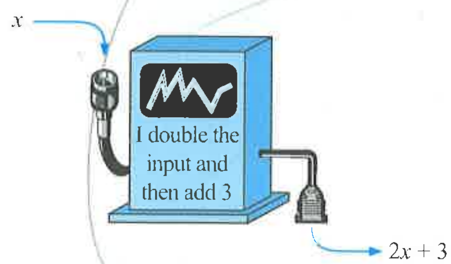
g $y = \dfrac{x+4}{x-2}$ **h** $y = x^3 - 3x^2 - 9x + 10$ **i** $y = \dfrac{3x-9}{x^2-x-2}$

j $y = x^2 + x^{-2}$ **k** $y = x^3 + \dfrac{1}{x^3}$ **l** $y = x^4 + 4x^3 - 16x + 3$

C FUNCTION NOTATION

Function machines are sometimes used to illustrate how functions behave.

For example:



So, if 4 is fed into the machine, $2(4) + 3 = 11$ comes out.

The above 'machine' has been programmed to perform a particular function.
If f is used to represent that particular function we can write:

f is the function that will convert x into $2x + 3$.

$f(4) = 2x+3$

Think of it as "F is the machine that will take x and make it into 2x+3.

So, f would convert 2 into $2(2) + 3 = 7$ and
-4 into $2(-4) + 3 = -5$.

This function can be written as:

$$f : x \longmapsto \underbrace{2x + 3}$$

function f such that x is converted into $2x + 3$

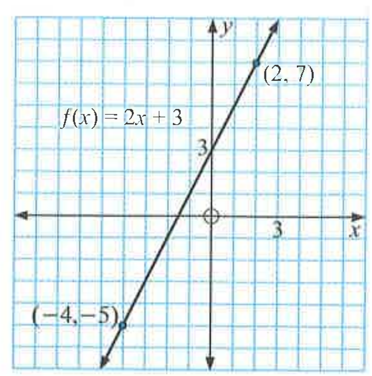
Two other equivalent forms we use are: $f(x) = 2x + 3$ or $y = 2x + 3$

So, $f(x)$ is the value of y for a given value of x, i.e., $y = f(x)$.

Notice that for $f(x) = 2x + 3$, $f(2) = 2(2) + 3 = 7$ and
$f(-4) = 2(-4) + 3 = -5$.

Consequently, $f(2) = 7$ indicates that the point $(2, 7)$ lies on the graph of the function.

Likewise $f(-4) = -5$ indicates that the point $(-4, -5)$ also lies on the graph.



Note:
- $f(x)$ is read as "f of x" and is the value of the function at any value of x.
- If (x, y) is any point on the graph then $y = f(x)$.
- f is the function which converts x into $f(x)$, i.e., $f : x \longmapsto f(x)$.
- $f(x)$ is sometimes called the **image** of x.

Example 3

If $f : x \longmapsto 2x^2 - 3x$, find the value of: **a** $f(5)$ **b** $f(-4)$

$f(x) = 2x^2 - 3x$

a $f(5) = 2(5)^2 - 3(5)$ {replacing x by (5)}
$= 2 \times 25 - 15$
$= 35$

b $f(-4) = 2(-4)^2 - 3(-4)$ {replacing x by (-4)}
$= 2(16) + 12$
$= 44$

EXERCISE 1C

1 If $f : x \longmapsto 3x + 2$, find the value of:
a $f(0)$ **b** $f(2)$ **c** $f(-1)$ **d** $f(-5)$ **e** $f(-\frac{1}{3})$

2 If $g : x \longmapsto x - \dfrac{4}{x}$, find the value of:
a $g(1)$ **b** $g(4)$ **c** $g(-1)$ **d** $g(-4)$ **e** $g(-\frac{1}{2})$

3 If $f : x \longmapsto 3x - x^2 + 2$, find the value of:
a $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(-7)$ **e** $f(\frac{3}{2})$

Example 4

If $f(x) = 5 - x - x^2$, find in simplest form: **a** $f(-x)$ **b** $f(x+2)$

a $f(-x) = 5 - (-x) - (-x)^2$ {replacing x by $(-x)$}
$= 5 + x - x^2$

b $f(x+2) = 5 - (x+2) - (x+2)^2$ {replacing x by $(x+2)$}
$= 5 - x - 2 - [x^2 + 4x + 4]$
$= 3 - x - x^2 - 4x - 4$
$= -x^2 - 5x - 1$

4 If $f(x) = 7 - 3x$, find in simplest form:
a $f(a)$ **b** $f(-a)$ **c** $f(a+3)$ **d** $f(b-1)$ **e** $f(x+2)$

5 If $F(x) = 2x^2 + 3x - 1$, find in simplest form:

a $F(x+4)$ **b** $F(2-x)$ **c** $F(-x)$ **d** $F(x^2)$ **e** $F(x^2-1)$

6 If $G(x) = \dfrac{2x+3}{x-4}$:

a evaluate **i** $G(2)$ **ii** $G(0)$ **iii** $G(-\frac{1}{2})$

b find a value of x where $G(x)$ does not exist

c find $G(x+2)$ in simplest form

d find x if $G(x) = -3$.

7 f represents a function. What is the difference in meaning between f and $f(x)$?

8 If $f(x) = 2^x$, show that $f(a)f(b) = f(a+b)$.

9 Given $f(x) = x^2$ find in simplest form:

a $\dfrac{f(x) - f(3)}{x-3}$ **b** $\dfrac{f(2+h) - f(2)}{h}$

10 If the value of a photocopier t years after purchase is given by $V(t) = 9650 - 860t$ dollars:

a find $V(4)$ and state what $V(4)$ means

b find t when $V(t) = 5780$ and explain what this represents

c find the original purchase price of the photocopier.

11 On the same set of axes draw the graphs of three different functions $f(x)$ such that $f(2) = 1$ and $f(5) = 3$.

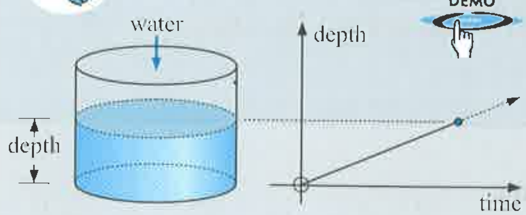
12 Find $f(x) = ax + b$, a linear function, in which $f(2) = 1$ and $f(-3) = 11$.

13 Find constants a and b where $f(x) = ax + \dfrac{b}{x}$ and $f(1) = 1$, $f(2) = 5$.

14 Given $T(x) = ax^2 + bx + c$, find a, b and c if $T(0) = -4$, $T(1) = -2$ and $T(2) = 6$.

INVESTIGATION — FLUID FILLING FUNCTIONS

When water is added at a **constant rate** to a cylindrical container the depth of water in the container is a function of time.

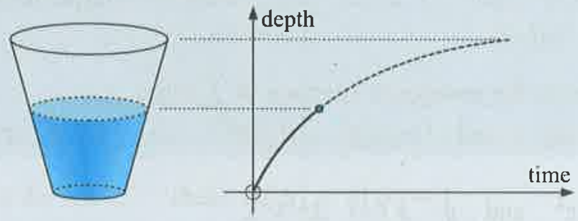


DEMO

This is because the volume of water added is directly proportional to the time taken to add it. If water was not added at a constant rate the direct proportionality would not exist.

The depth-time graph for the case of a cylinder would be as shown alongside.

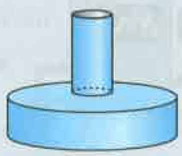
The question arises: 'What changes in appearance of the graph occur for different shaped containers?' Consider a vase of conical shape.

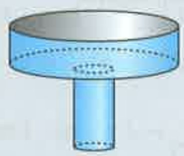


DEMO

What to do:

1. For each of the following containers, draw a 'depth v time' graph as water is added:

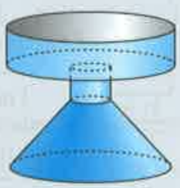
a


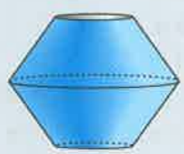
b


c


d

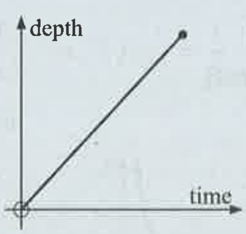

e

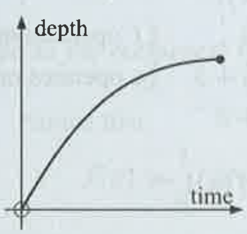

f


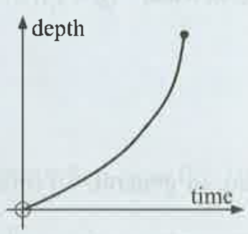
g


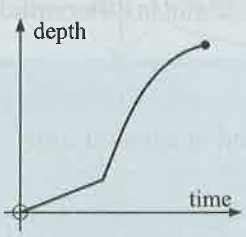
h

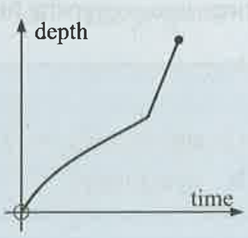

2. Use the water filling demonstration to check your answers to question **1**.
3. Write a brief report on the connection between the shape of a vessel and the corresponding shape of its depth-time graph. You may wish to discuss this in parts. For example, first examine cylindrical containers, then conical, then other shapes. Slopes of curves must be included in your report.
4. Draw possible containers as in question **1** which have the following 'depth v time' graphs:

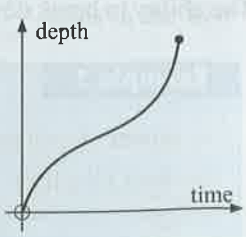
a


b


c


d


e


f


D COMPOSITE FUNCTIONS, $f \circ g$

Given $f : x \longmapsto f(x)$ and $g : x \longmapsto g(x)$, then the **composite function** of f and g will convert x into $f(g(x))$.

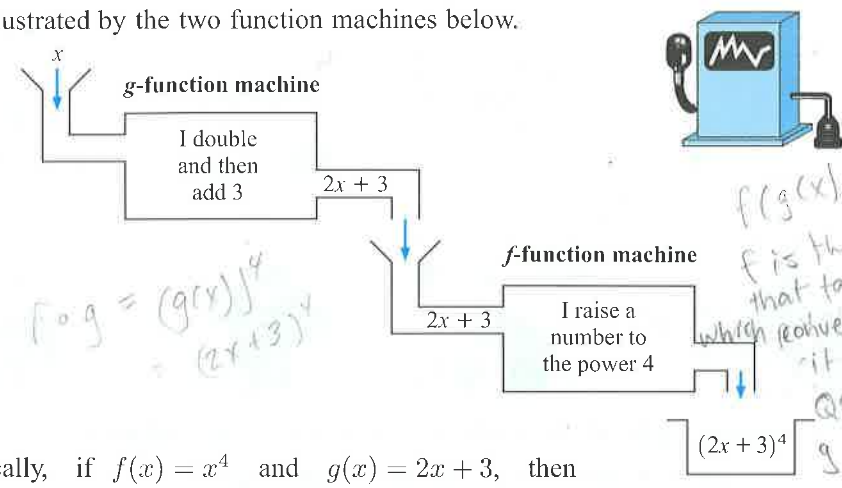
$f \circ g$ is used to represent the composite function of f and g.
$f \circ g$ means f following g and $(f \circ g)(x) = f(g(x))$, i.e., $f \circ g : x \longmapsto f(g(x))$.

Consider $f : x \longmapsto x^4$ and $g : x \longmapsto 2x + 3$.

DEMO

$f \circ g$ means that g converts x to $2x + 3$ and then f converts $(2x + 3)$ to $(2x + 3)^4$.

This is illustrated by the two function machines below.



$f \circ g = (g(x))^4$
$= (2x+3)^4$

$f(g(x))$
f is the machine that takes the machine, g, which converts x ~~and puts~~ it to the power of 4.
Q: what did the g machine do to x?
$2x+3$

Algebraically, if $f(x) = x^4$ and $g(x) = 2x + 3$, then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x+3) \quad \{g \text{ operates on } x \text{ first}\} \\ &= (2x+3)^4 \quad \{f \text{ operates on } g(x) \text{ next}\}\end{aligned}$$

$f(g(x)) = g(x)^4$
$= (2x+3)^4$

Likewise,
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^4) \quad \{f \text{ operates on } x \text{ first}\} \\ &= 2(x^4) + 3 \quad \{g \text{ operates on } f(x) \text{ next}\} \\ &= 2x^4 + 3\end{aligned}$$

So, in general, $f(g(x)) \neq g(f(x))$.

The ability to break down functions into composite functions is useful in **differential calculus**.

Example 5

Given $f : x \longmapsto 2x + 1$ and $g : x \longmapsto 3 - 4x$ find in simplest form:
a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

$f(x) = 2x+1$ and $g(x) = 3-4x$

a $\therefore\ (f \circ g)(x) = f(g(x))$
$= f(3-4x)$
$= 2(3-4x)+1$
$= 6-8x+1$
$= 7-8x$

b $(g \circ f)(x) = g(f(x))$
$= g(2x+1)$
$= 3-4(2x+1)$
$= 3-8x-4$
$= -8x-1$

Note: If $f(x) = 2x+1$ then $f(\Delta) = 2(\Delta)+1$
$f(*) = 2(*)+1$
and $f(3-4x) = 2(3-4x)+1$

EXERCISE 1D

1 Given $f : x \mapsto 2x+3$ and $g : x \mapsto 1-x$, find in simplest form:
a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(f \circ g)(-3)$

2 Given $f : x \mapsto x^2$ and $g : x \mapsto 2-x$ find $(f \circ g)(x)$ and $(g \circ f)(x)$.

3 Given $f : x \mapsto x^2+1$ and $g : x \mapsto 3-x$, find in simplest form:
a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** x if $(g \circ f)(x) = f(x)$

4 **a** If $ax+b = cx+d$ for all values of x, show that $a = c$ and $b = d$.
(**Hint:** If it is true for all x, it is true for $x = 0$ and $x = 1$.)

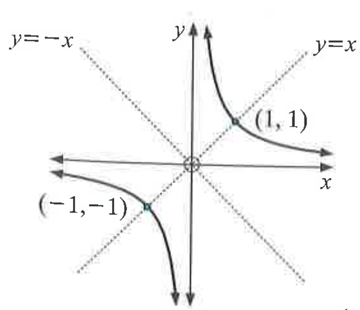
b Given $f(x) = 2x+3$ and $g(x) = ax+b$ and that $(f \circ g)(x) = x$ for all values of x, deduce that $a = \frac{1}{2}$ and $b = -\frac{3}{2}$.

c Is the result in **b** true if $(g \circ f)(x) = x$ for all x?

E THE RECIPROCAL FUNCTION $x \mapsto \frac{1}{x}$

$x \mapsto \frac{1}{x}$, i.e., $f(x) = \frac{1}{x}$ is defined as the **reciprocal function**.

It has graph:



The two branches of $y = \frac{1}{x}$

Notice that:

- $f(x) = \frac{1}{x}$ is meaningless when $x = 0$
- The graph of $f(x) = \frac{1}{x}$ exists in the first and third quadrants only.
- $f(x) = \frac{1}{x}$ is symmetric about $y = x$ and $y = -x$

GRAPHING PACKAGE

- $f(x) = \dfrac{1}{x}$ is **asymptotic** (approaches) to the x-axis and to the y-axis.

- as $x \to \infty$, $f(x) \to 0$ (above)
  as $x \to -\infty$, $f(x) \to 0$ (below)
  as $y \to \infty$, $x \to 0$ (right)
  as $y \to -\infty$, $x \to 0$ (left)
  $\to$ reads *approaches* or *tends to*

EXERCISE 1E

1 Sketch the graph of $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{2}{x}$, $h(x) = \dfrac{4}{x}$ on the same set of axes. Comment on any similarities and differences.

2 Sketch the graphs of $f(x) = -\dfrac{1}{x}$, $g(x) = -\dfrac{2}{x}$, $h(x) = -\dfrac{4}{x}$ on the same set of axes. Comment on any similarities and differences.

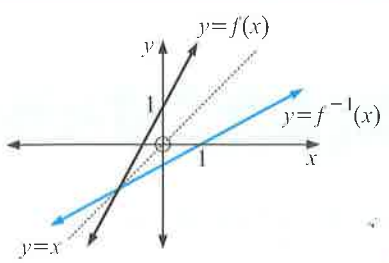
F INVERSE FUNCTIONS

A function $y = f(x)$ *may or may not* have an inverse function.

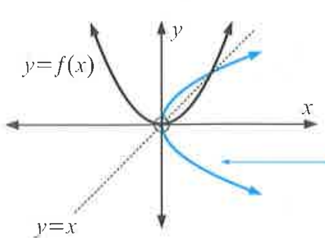
If $y = f(x)$ has an **inverse function**, this new function
- must indeed be a function, i.e., satisfy the vertical line test and it
- must be the reflection of $y = f(x)$ in the line $y = x$.

The inverse function of $y = f(x)$ is denoted by $y = f^{-1}(x)$.
If (x, y) lies on f, then (y, x) lies on f^{-1}. So reflecting the function in $y = x$ has the algebraic effect of interchanging x and y,
e.g., $f : y = 5x + 2$ becomes $f^{-1} : x = 5y + 2$.

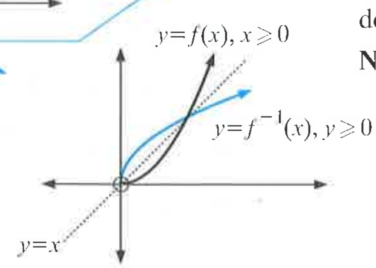
For example,



$y = f^{-1}(x)$ is the inverse of $y = f(x)$ as
- it is also a function
- it is the reflection of $y = f(x)$ in the oblique line $y = x$.



This is the reflection of $y = f(x)$ in $y = x$, but it is not the inverse function of $y = f(x)$ as it fails the vertical line test.
We say that the function $y = f(x)$ does not have an inverse.



Note: $y = f(x)$ subject to $x \geqslant 0$ does have an inverse function.

Also, $y = f(x)$ subject to $x \leqslant 0$ does have an inverse function.

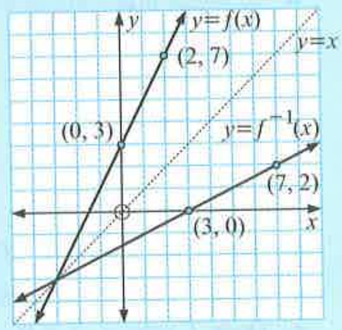
Example 6

Consider $f : x \longmapsto 2x+3$.

a On the same axes, graph f and its inverse function f^{-1}.

b Find $f^{-1}(x)$ using **i** coordinate geometry and the slope of $f^{-1}(x)$ from **a**
ii variable interchange.

a $f(x) = 2x+3$ passes through (0, 3) and (2, 7).
$\therefore$ $f^{-1}(x)$ passes through (3, 0) and (7, 2).



b **i** This line has slope $\dfrac{2-0}{7-3} = \dfrac{1}{2}$.

So, its equation is

$$\frac{y-0}{x-3} = \frac{1}{2}$$

i.e., $y = \dfrac{x-3}{2}$

i.e., $f^{-1}(x) = \dfrac{x-3}{2}$

ii f is $y = 2x+3$, so f^{-1} is $x = 2y+3$

$\therefore$ $x - 3 = 2y$

$\therefore$ $\dfrac{x-3}{2} = y$ i.e., $f^{-1}(x) = \dfrac{x-3}{2}$

Note: If f includes point (a, b) then f^{-1} includes point (b, a), i.e., the point obtained by interchanging the coordinates.

EXERCISE 1F

1 Consider $f : x \longmapsto 3x+1$.

a On the same axes graph $y = x$, f and f^{-1}.

b Find $f^{-1}(x)$ using coordinate geometry and **a**.

c Find $f^{-1}(x)$ using variable interchange.

2 Consider $f : x \longmapsto \dfrac{x+2}{4}$.

a On the same set of axes graph $y = x$, f and f^{-1}.

b Find $f^{-1}(x)$ using coordinate geometry and **a**.

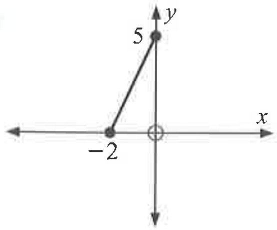
c Find $f^{-1}(x)$ using variable interchange.

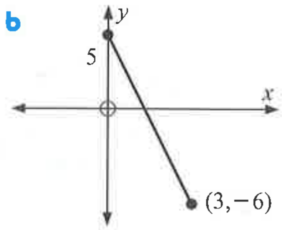
3 For each of the following functions f

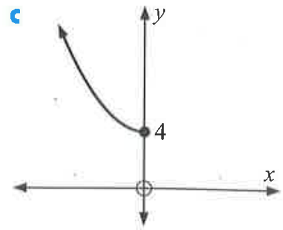
i find $f^{-1}(x)$ **ii** sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same axes:

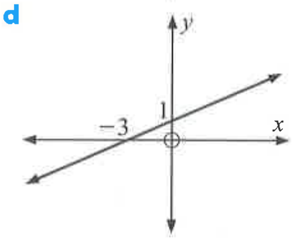
a $f : x \longmapsto 2x+5$ **b** $f : x \longmapsto \dfrac{3-2x}{4}$ **c** $f : x \longmapsto x+3$

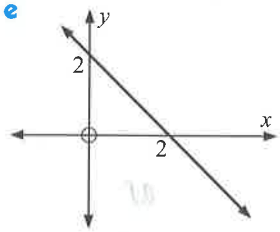
4 Copy the graphs of the following functions and in each case include the graphs of $y = x$ and $y = f^{-1}(x)$.

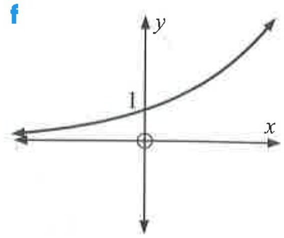
a


b


c


d


e


f


5 **a** Sketch the graph of $f : x \longmapsto x^2 - 4$ and reflect it in the line $y = x$.

b Does f have an inverse function?

c Does f where $x \geqslant 0$ have an inverse function?

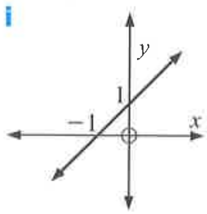
6 Sketch the graph of $f : x \longmapsto x^3$ and its inverse function $f^{-1}(x)$.

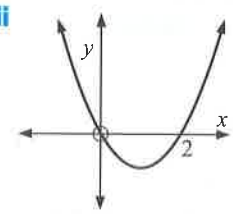
7 The '**horizontal line test**' says that:

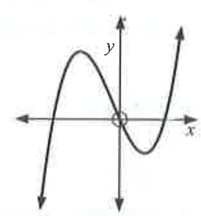
for a function to have an inverse function, no horizontal line can cut it more than once.

a Explain why this is a valid test for the existence of an inverse function.

b Which of the following functions have an inverse function?

i


ii


iii


Example 7

Consider $f : x \longmapsto x^2$ where $x \geqslant 0$.

a Find $f^{-1}(x)$.

b Sketch $y = f(x)$, $y = x$ and $y = f^{-1}(x)$ on the same set of axes.

---

a f is defined by $y = x^2,\ x \geqslant 0$

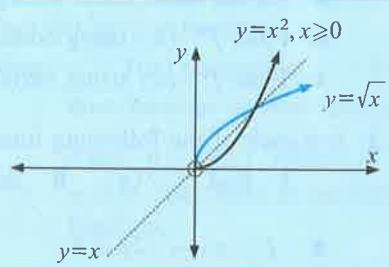
$\therefore$ f^{-1} is defined by $x = y^2,\ y \geqslant 0$

$\therefore$ $y = \pm\sqrt{x},\ y \geqslant 0$

i.e., $y = \sqrt{x}$

{as $-\sqrt{x}$ is $\leqslant 0$}

So, $f^{-1}(x) = \sqrt{x}$

b


8 Consider $f : x \longmapsto x^2$ where $x \leqslant 0$.

- **a** Find $f^{-1}(x)$.
- **b** Sketch $y = f(x)$, $y = x$ and $y = f^{-1}(x)$ on the same set of axes.

9
- **a** Explain why $f : x \longmapsto x^2 - 4x + 3$ is a function but does not have an inverse function.
- **b** Explain why f for $x \geqslant 2$ has an inverse function.
- **c** Show that the inverse function of the function in **b** is $f^{-1}(x) = 2 + \sqrt{1+x}$.
- **d** If the domain of f is restricted to $x \geqslant 2$, state the domain and range of
  i f **ii** f^{-1}.

10 Consider $f(x) = \frac{1}{2}x - 1$.

- **a** Find $f^{-1}(x)$.
- **b** Find **i** $(f \circ f^{-1})(x)$ **ii** $(f^{-1} \circ f)(x)$.

11 Given $f : x \longmapsto (x+1)^2 + 3$ where $x \geqslant -1$,

- **a** find the defining equation of f^{-1}
- **b** sketch, using technology, the graphs of $y = f(x)$, $y = x$ and $y = f^{-1}(x)$
- **c** state the domain and range of **i** f **ii** f^{-1}.

12 Consider the functions $f : x \longmapsto 2x + 5$ and $g : x \longmapsto \dfrac{8-x}{2}$.

- **a** Find $g^{-1}(-1)$.
- **b** Solve for x the equation $(f \circ g^{-1})(x) = 9$.

13 Given $f : x \longmapsto 5^x$ and $g : x \longmapsto \sqrt{x}$,

- **a** find **i** $f(2)$ **ii** $g^{-1}(4)$
- **b** solve the equation $(g^{-1} \circ f)(x) = 25$.

14 Given $f : x \longmapsto 2x$ and $g : x \longmapsto 4x - 3$ show that
$(f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$.

15 Which of these functions are their own inverses, that is $f^{-1}(x) = f(x)$?

a $f(x) = 2x$ **b** $f(x) = x$ **c** $f(x) = -x$ **d** $f(x) = \dfrac{1}{x}$ **e** $f(x) = -\dfrac{6}{x}$

G THE IDENTITY FUNCTION

In question **10** of the previous exercise we considered $f(x) = \frac{1}{2}x - 1$.

We found that $f^{-1}(x) = 2x + 2$ and that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

$e(x) = x$ is called the **identity function** of function $y = f(x)$
It is the unique solution of $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = e(x)$.

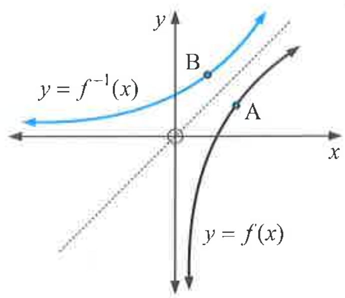
EXERCISE 1G

1 For $f(x) = 3x + 1$, find $f^{-1}(x)$ and show that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

2 For $f(x) = \dfrac{x+3}{4}$, find $f^{-1}(x)$ and show that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

3 For $f(x) = \sqrt{x}$, find $f^{-1}(x)$ and show that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

4



a B is the image of A under a reflection in the line $y = x$.
If A is $(x, f(x))$, what are the coordinates of B under the reflection?

b Substitute your result from **a** into $y = f^{-1}(x)$.
What result do you obtain?

c Explain how to establish that $f(f^{-1}(x)) = x$ also.

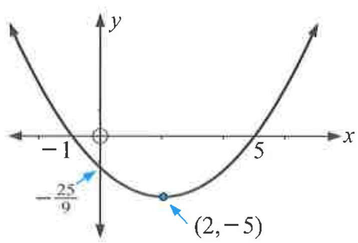
REVIEW SET 1A

1 Draw a graph to show what happens in the following jar-water-golf ball situation:
Water is added to an empty jar at a constant rate for two minutes and then one golf ball is added. After one minute another golf ball is added. Two minutes later both golf balls are removed. Half the water is then removed at a constant rate over a two minute period.

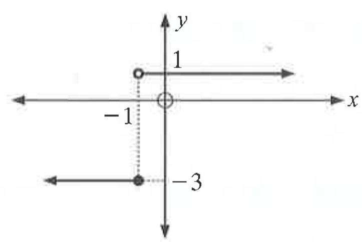
2 If $f(x) = 2x - x^2$ find: **a** $f(2)$ **b** $f(-3)$ **c** $f(-\frac{1}{2})$

3 For the following graphs determine:
i the range and domain **ii** the x and y-intercepts **iii** whether it is a function.

a

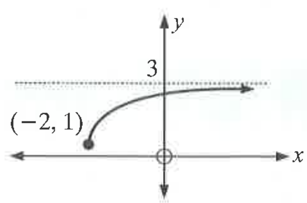


b

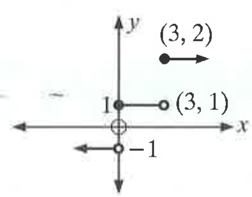


4 For each of the following graphs find the domain and range:

a



b



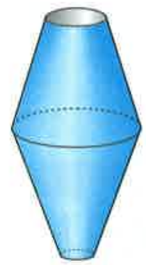
5 If $h(x) = 7 - 3x$:
a find in simplest form $h(2x - 1)$ **b** find x if $h(2x-1) = -2$

6 If $f(x) = ax + b$ where a and b are constants, find a and b for $f(1) = 7$ and $f(3) = -5$.

7 Find a, b and c if $f(0) = 5$, $f(-2) = 21$ and $f(3) = -4$ for $f(x) = ax^2 + bx + c$.

8 For each of the following containers draw a 'depth v time' graph as water is added.

a



b

9 Consider $f(x) = \dfrac{1}{x^2}$.

a For what value of x is $f(x)$ meaningless?
b Sketch the graph of this function using technology.
c State the domain and range of the function.

10 If $f(x) = 2x - 3$ and $g(x) = x^2 + 2$, find in simplest form:

a $f(g(x))$ **b** $g(f(x))$

11 If $f(x) = 1 - 2x$ and $g(x) = \sqrt{x}$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

12 Find an f and a g function given that:

a $f(g(x)) = \sqrt{1 - x^2}$ **b** $g(f(x)) = \left(\dfrac{x-2}{x+1}\right)^2$

REVIEW SET 1B

1 If $f(x) = 5 - 2x$, find **a** $f(0)$ **b** $f(5)$ **c** $f(-3)$ **d** $f(\frac{1}{2})$

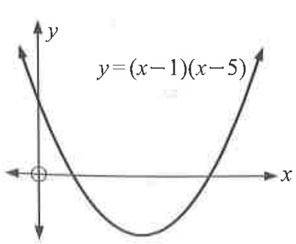
2 If $g(x) = x^2 - 3x$, find in simplest form **a** $g(x+1)$ **b** $g(x^2 - 2)$

3 For each of the following functions $f(x)$ find $f^{-1}(x)$:

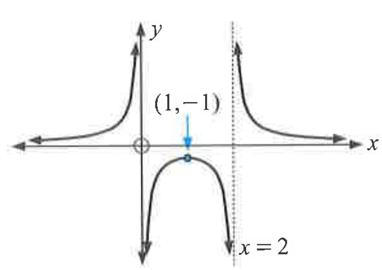
a $f(x) = 7 - 4x$ **b** $f(x) = \dfrac{3 + 2x}{5}$

4 For each of the following graphs, find the domain and range.

a

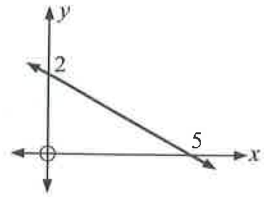


b

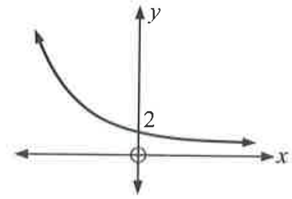


5 Copy the following graphs and draw the graph of each inverse function:

a



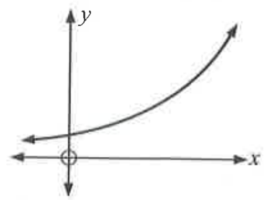
b



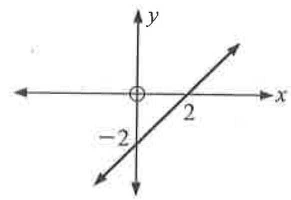
6 Find $f^{-1}(x)$ given that $f(x)$ is: **a** $4x+2$ **b** $\dfrac{3-5x}{4}$

7 Copy the following graphs and draw the graph of each inverse function:

a



b



8 Given $f(x) = 2x + 11$ and $g(x) = x^2$, find $(g \circ f^{-1})(3)$.

9 Consider $x \longmapsto 2x - 7$.

a On the same set of axes graph $y = x$, f and f^{-1}.

b Find $f^{-1}(x)$ using coordinate geometry.

c Find $f^{-1}(x)$ using variable interchange.

10 **a** Sketch the graph of $g : x \longmapsto x^2 + 6x + 7$.

b Explain why g for $x \leqslant -3$ has an inverse function g^{-1}.

c Find algebraically, the equation of g^{-1}.

d Sketch the graph of g^{-1}.

11 Given $h : x \longmapsto (x-4)^2 + 3$ where $x \geqslant 4$, find the defining equation of h^{-1}.

12 Given $f : x \longmapsto 3x + 6$ and $h : x \longmapsto \dfrac{x}{3}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.